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Journal of Algebra 310 (2007) 452–457
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Karoubianness of a triangulated category ☆

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Received 24 October 2006
Available online 4 January 2007
Communicated by Michel Van den Bergh

Dedicated to Professor Yingbo Zhang on the occasion of her sixtieth birthday

Abstract

We prove that triangulated categories with bounded t-structures are Karoubian. Consequently, for an Ext-finite abelian category over a commutative noetherian complete local ring, its bounded derived category is Krull–Schmidt.

Keywords: Karoubianness; Triangulated category; t-Structure

1. Introduction

Let $\mathcal{D}$ be a triangulated category [V] with its shift functor denoted by [1]. Recall from [BBD] that a t-*structure* on $\mathcal{D}$ is a pair of strictly (i.e. closed under isomorphisms) full additive subcategories $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ satisfying the following conditions:

(T1) $\mathrm{Hom}_{\mathcal{D}}(X, Y[-1]) = 0$ for all $X \in \mathcal{D}^{\leqslant 0}$ and $Y \in \mathcal{D}^{\geqslant 0}$;
(T2) $D^{\leqslant 0}$ is closed under the functor [1], and $D^{\geqslant 0}$ is closed under the functor $[-1]$;
(T3) for each $X \in \mathcal{D}$, there is an exact triangle $A \to X \to B[-1] \to A[1]$ with $A \in \mathcal{D}^{\leqslant 0}$ and $B \in \mathcal{D}^{\geqslant 0}$.

☆ Partly supported by the National Natural Science Foundation of China (Grant Nos. 10301033 and 10501041).
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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2006.11.027

Set $\mathcal{D}^{\leqslant n} = \mathcal{D}^{\leqslant 0}[-n]$ and $\mathcal{D}^{\geqslant n} = \mathcal{D}^{\geqslant 0}[-n]$, $n \in \mathbb{Z}$. The t-structure is called *bounded* (cf. [GM1, p. 136] and [GM2, p. 286, Exercises]) if for each $X \in \mathcal{D}$, there exists $m \leqslant n$ such that $X \in \mathcal{D}^{\leqslant n} \cap \mathcal{D}^{\geqslant m}$.

Recall that in an additive category $\mathfrak{a}$, an idempotent morphism $e : X \to X$ is said to be *split* if there are two morphisms $u : X \to Y$ and $v : Y \to X$ such that $v \circ u = e$ and $u \circ v = \mathrm{Id}_Y$. The category $\mathfrak{a}$ is said to be *Karoubian* (i.e. *idempotent-split*) provided that every idempotent-splits.

Our main theorem is

Theorem. *Let $\mathcal{D}$ be a triangulated category with a bounded t-structure. Then $\mathcal{D}$ is Karoubian.*

Let $\mathcal{A}$ be an abelian category. It is well known that the bounded derived category $D^b(\mathcal{A})$ has a natural bounded t-structure. So we have

Corollary A. [BS, Corollary 2.10] *Let $\mathcal{A}$ be an abelian category. Then the bounded derived category $D^b(\mathcal{A})$ is Karoubian.*

Let R be a commutative noetherian ring which is complete and local. An abelian category $\mathcal{A}$ over R is said to be *Ext-finite*, if for each $X, Y \in \mathcal{A}$, $n \geqslant 0$, the R-module $\mathrm{Ext}^n_{\mathcal{A}}(X, Y)$ is finitely-generated. It is not hard to see that $\mathcal{A}$ is Ext-finite if and only if $D^b(\mathcal{A})$ is Hom-finite over R. Recall that an additive category is *Krull–Schmidt* if each object is a finite direct sum of indecomposables with local endomorphism rings. It is shown in [CYZ, Theorem A.1] that an additive category is Krull–Schmidt if and only if it is Karoubian and for each object X, $\mathrm{End}(X)$ is a semiperfect ring. Finally note that an algebra over R which is finitely-generated as a R-module is semiperfect (cf. [L, Example (23.3)]). So we have

Corollary B. *Let R be a commutative noetherian ring which is complete and local, and let $\mathcal{A}$ be an Ext-finite abelian category over R. Then the bounded derived category $D^b(\mathcal{A})$ is a Krull–Schmidt category.*

2. Proof of Theorem

Before proving the theorem, we need some preparations.

2.1. Let $\mathcal{C}$ be a triangulated category. The following lemma is well known.

Lemma 2.1. *Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ be an exact triangle. Then we have*

(1) *If $e : Z \to Z$ is a morphism satisfying $e \circ v = v$ and $w \circ e = w$, then e is an isomorphism.*
(2) *Assume that $x : Z \to Z'$ and $y : Z' \to Z$ are two morphisms satisfying $x \circ v = 0$ and $w \circ y = 0$. Then $x \circ y = 0$.*

Proof. (1) By assumption, we have the following morphism of exact triangles

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ \| & & \| & & \downarrow{\scriptstyle e} & & \| \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1]. \end{array}$$

Then it is well known that e is an isomorphism (e.g., by using [GM2, IV.1, Corollary 4(a)]).

(2) Since $x \circ v = 0$, then it is again well known that x factors through w (e.g., by using [GM2, IV.1, Proposition 3]). Suppose $x' : X[1] \to Z'$ such that $x = x' \circ w$. Hence $x \circ y = x' \circ w \circ y = 0$. $\square$

Let $\mathfrak{a}$ be any additive category. An idempotent $e : X \to X$ splits if there are morphisms $u : X \to Y$ and $v : Y \to X$ such that $v \circ u = e$ and $u \circ v = \mathrm{Id}_Y$. Then u and v are the cokernel and kernel of the morphism $\mathrm{Id}_X - e$, respectively. Moreover, it is not hard to see that an idempotent e splits if and only if $1 - e$ has a kernel, if and only if $1 - e$ has a cokernel. We say that an idempotent e *strongly splits* if both e and $1 - e$ split. In this case, assume that $1 - e$ splits as $X \xrightarrow{u'} Y' \xrightarrow{v'} X$, then $\binom{u}{u'} : X \to Y \oplus Y'$ is an isomorphism, whose inverse is given by $(v\ v')$.

The following lemma seems to be known.

Lemma 2.2. *Let $e : X \to X$ be an idempotent morphism in a triangulated category $\mathcal{C}$. Then e splits if and only if e strongly splits.*

Proof. We just prove the "only if" part. Assume that e splits as $X \xrightarrow{u} Y \xrightarrow{v} X$. We need to prove that $1 - e$ splits. By the above facts, it suffices to show that e has a cokernel. Since $e = v \circ u$ and that u is clearly epi, thus we know the cokernel of v, if in existence, is just the cokernel of e.

Take an exact triangle $Y \xrightarrow{v} X \xrightarrow{\pi} Z \to Y[1]$. Note that v is a section, then by [H, p. 7, Lemma 1.4] one obtains that π is a retraction. Now using [H, Chapter I, Proposition 1.2(b)], it is not hard to see that π is the cokernel of v, and thus the cokernel of e. This completes the proof. $\square$

We have the following key observation.

Proposition 2.3. *Let the following diagram be a morphism of exact triangles*

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\
\downarrow{\scriptstyle e_1} & & \downarrow{\scriptstyle e_2} & & \downarrow{\scriptstyle e_3} & & \downarrow{\scriptstyle e_1[1]} \\
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1]
\end{array}$$

with each e_i an idempotent. Then if e_1 and e_2 splits, so does e_3.

Proof. By Lemma 2.2, both e_1 and e_2 strongly split. We may assume that $X = X_1 \oplus X_2$, $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $Y = Y_1 \oplus Y_2$, $e_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. By $e_2 \circ u = u \circ e_1$, one deduces that u is diagonalizable, say $u = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}$. Take the following exact triangles in $\mathcal{C}$:

$$X_i \xrightarrow{u_i} Y_i \xrightarrow{v_i} Z_i \xrightarrow{w_i} X_i[1], \quad i = 1, 2.$$

Hence there is an isomorphism of exact triangles

$$\begin{tikzcd}[column sep=large]
X_1 \oplus X_2 \arrow[r, "\left(\begin{smallmatrix} u_1 & 0 \\ 0 & u_2 \end{smallmatrix}\right)"] \arrow[d, equal] & Y_1 \oplus Y_2 \arrow[r, "\left(\begin{smallmatrix} v_1 & 0 \\ 0 & v_2 \end{smallmatrix}\right)"] \arrow[d, equal] & Z_1 \oplus Z_2 \arrow[r, "\left(\begin{smallmatrix} w_1 & 0 \\ 0 & w_2 \end{smallmatrix}\right)"] \arrow[d, "\theta"] & (X_1 \oplus X_2)[1] \arrow[d, equal] \\
X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r, "w"] & X[1].
\end{tikzcd}$$

Set $e = \theta^{-1} \circ e_3 \circ \theta$. Note that e is also an idempotent, and e splits if and only if e_3 splits. We have the following morphism of exact triangles

$$\begin{tikzcd}[column sep=large]
X_1 \oplus X_2 \arrow[r, "\left(\begin{smallmatrix} u_1 & 0 \\ 0 & u_2 \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)"'] & Y_1 \oplus Y_2 \arrow[r, "\left(\begin{smallmatrix} v_1 & 0 \\ 0 & v_2 \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)"] & Z_1 \oplus Z_2 \arrow[r, "\left(\begin{smallmatrix} w_1 & 0 \\ 0 & w_2 \end{smallmatrix}\right)"] \arrow[d, "e"] & (X_1 \oplus X_2)[1] \arrow[d, "\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)"] \\
X_1 \oplus X_2 \arrow[r, "\left(\begin{smallmatrix} u_1 & 0 \\ 0 & u_2 \end{smallmatrix}\right)"] & Y_1 \oplus Y_2 \arrow[r, "\left(\begin{smallmatrix} v_1 & 0 \\ 0 & v_2 \end{smallmatrix}\right)"] & Z_1 \oplus Z_2 \arrow[r, "\left(\begin{smallmatrix} w_1 & 0 \\ 0 & w_2 \end{smallmatrix}\right)"] & (X_1 \oplus X_2)[1].
\end{tikzcd}$$

Write $e = \left(\begin{smallmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{smallmatrix}\right)$ in a matrix form. By the commutativity of the diagram (and using some matrix calculation), we get

$$\begin{aligned}
&e_{11} \circ v_1 = v_1, \qquad && w_1 \circ e_{11} = w_1; && \\
&e_{12} \circ v_2 = 0, && e_{21} \circ v_1 = 0, \qquad && e_{22} \circ v_2 = 0; \\
&w_1 \circ e_{12} = 0, && w_2 \circ e_{21} = 0, && w_2 \circ e_{22} = 0.
\end{aligned}$$

Using Lemma 2.1(1), we deduce that e_{11} is an isomorphism. Applying Lemma 2.1(2) four times, we get

$$e_{12} \circ e_{21} = 0, \qquad e_{12} \circ e_{22} = 0, \qquad e_{22}^2 = 0 \quad \text{and} \quad e_{21} \circ e_{12} = 0.$$

By $e^2 = e$ and using the above four identities, we obtain

$$e_{11}^2 = e_{11}, \qquad e_{11} \circ e_{12} = e_{12}, \qquad e_{21} \circ e_{11} + e_{22} \circ e_{21} = e_{21} \quad \text{and} \quad e_{22} = 0.$$

Then $e_{11} = \mathrm{Id}_{Z_1}$ and $e = \left(\begin{smallmatrix} 1 & e_{12} \\ e_{21} & 0 \end{smallmatrix}\right)$. Note that $e_{12} \circ e_{21} = 0$ and $e_{21} \circ e_{12} = 0$, hence e splits as

$$Z_1 \oplus Z_2 \xrightarrow{(1\ e_{12})} Z_1 \xrightarrow{\left(\begin{smallmatrix} 1 \\ e_{21} \end{smallmatrix}\right)} Z_1 \oplus Z_2.$$

This completes the proof. $\square$

Remark 2.4. Note that the proofs of Lemmas 2.1, 2.2 and Proposition 2.3 do not use the axiom (TR4) in [V, p. 3]. Hence they hold for pre-triangulated categories.

2.2. In what follows, $\mathcal{D}$ is a triangulated category with a t-structure $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$. By [H, p. 58], the pair $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 1})$ is a *torsion pair* of $\mathcal{D}$, in particular, $\mathcal{D}^{\leqslant 0}$ and $\mathcal{D}^{\geqslant 1}$ are closed under "extensions," i.e., for any exact triangle $X \to Y \to Z \to X[1]$ with $X, Z \in \mathcal{D}^{\leqslant 0}$ (respectively $X, Z \in \mathcal{D}^{\geqslant 1}$), so does Y. Now it is not hard to infer that both $\mathcal{D}^{\leqslant n}$ and $\mathcal{D}^{\geqslant n}$ are closed under extensions for each $n \in \mathbb{Z}$.

Recall from [GM1, p. 134] and [GM2, IV.4] there are *truncation functors* $\tau_{\leqslant 0} : \mathcal{D} \to \mathcal{D}^{\leqslant 0}$ and $\tau_{\geqslant 1} : \mathcal{D} \to \mathcal{D}^{\geqslant 1}$ which satisfy the following conditions:

(1) for each $X \in \mathcal{D}$, there is an exact triangle $\tau_{\leqslant 0}X \to X \to \tau_{\geqslant 1}X \to (\tau_{\leqslant 0}X)[1]$ (cf. axiom (T3));
(2) for each morphism $f : X \to Y$, one has the following morphism of triangles

$$\begin{tikzcd}
\tau_{\leqslant 0}X \arrow[r] \arrow[d, "\tau_{\leqslant 0}(f)"] & X \arrow[r] \arrow[d, "f"] & \tau_{\geqslant 1}X \arrow[r] \arrow[d, "\tau_{\geqslant 1}(f)"] & (\tau_{\leqslant 0}X)[1] \arrow[d, "\tau_{\leqslant 0}(f)[1]"] \\
\tau_{\leqslant 0}Y \arrow[r] & Y \arrow[r] & \tau_{\geqslant 1}Y \arrow[r] & (\tau_{\leqslant 0}Y)[1].
\end{tikzcd}$$

In general, we define $\tau_{\leqslant n} : \mathcal{D} \to \mathcal{D}^{\leqslant n}$ and $\tau_{\geqslant n+1} : \mathcal{D} \to \mathcal{D}^{\geqslant n+1}$ by $\tau_{\leqslant n} = [-n] \circ \tau_{\leqslant 0} \circ [n]$ and $\tau_{\geqslant n+1} = [-n] \circ \tau_{\geqslant 1} \circ [n]$, respectively. Then it is not hard to see that similar conditions as (1) and (2) hold for $\tau_{\leqslant n}$ and $\tau_{\geqslant n+1}$.

The following fact is easy (cf. [GM2, p. 280]).

Lemma 2.5. *Let $m \leqslant n$. Then $\tau_{\leqslant n}(\mathcal{D}^{\geqslant m}) \subseteq \mathcal{D}^{\geqslant m} \cap \mathcal{D}^{\leqslant n}$ and $\tau_{\geqslant m}(\mathcal{D}^{\leqslant n}) \subseteq \mathcal{D}^{\leqslant n} \cap \mathcal{D}^{\geqslant m}$.*

Proof. We only show the first inclusion. It suffices to show $\tau_{\leqslant n}(\mathcal{D}^{\geqslant m}) \subseteq \mathcal{D}^{\geqslant m}$. Let $X \in \mathcal{D}^{\geqslant m}$. Consider the exact triangle $\tau_{\leqslant n}X \to X \to \tau_{\geqslant n+1}X \to (\tau_{\leqslant n}X)[1]$. So $\tau_{\leqslant n}X$ is an extension of $(\tau_{\geqslant n+1}X)[-1]$ and X, both of which are easily seen to lie in $\mathcal{D}^{\geqslant m}$. Note that $\mathcal{D}^{\geqslant m}$ is closed under extensions, thus we infer that $X \in \mathcal{D}^{\geqslant m}$. $\square$

From now on, we assume that the t-structure in our consideration is bounded, i.e., for each X, there exists $m \leqslant n$ such that $X \in \mathcal{D}^{\leqslant n} \cap \mathcal{D}^{\geqslant m}$. First we note that $\bigcap_{n \in \mathbb{Z}} \mathcal{D}^{\leqslant n} = \{0\}$. To see this, let $X \in \bigcap_{n \in \mathbb{Z}} \mathcal{D}^{\leqslant n}$. By the bounded property, we may assume that $X \in \mathcal{D}^{\geqslant m+1}$ for some m. Note that $\mathrm{Hom}_{\mathcal{D}}(\mathcal{D}^{\leqslant m}, \mathcal{D}^{\geqslant m+1}) = 0$, and $X \in \mathcal{D}^{\leqslant m}$. So $\mathrm{Hom}_{\mathcal{D}}(X, X) = 0$, i.e., $X = 0$. Similarly, we have $\bigcap_{n \in \mathbb{Z}} \mathcal{D}^{\geqslant n} = \{0\}$. Therefore, the t-structure is *non-degenerate* in the sense of [GM1, p. 135, Theorem 3.5.1]. Moreover, by [GM1, p. 135, Theorem 3.5.1c] one sees immediately that our notion of bounded t-structures coincides with the one in [GM1, p. 136] (and also in [GM2, p. 286, Exercises]).

Let $X \in \mathcal{D}$ be non-zero. Set $b(X) = \max\{n \mid X \in \mathcal{D}^{\geqslant n}\}$, $t(X) = \min\{n \mid X \in \mathcal{D}^{\leqslant n}\}$ and $w(X) = t(X) - b(X) + 1$. If X is zero, set $w(X) = 0$. By the above non-degeneratedness, we know that $b(X)$ and $t(X)$ are well defined. It is direct to see that $w(X) \geqslant 0$, which will be called the *width* of X.

Proof of Theorem. Set $\mathcal{A} = \mathcal{D}^{\leqslant 0} \cap \mathcal{D}^{\geqslant 0}$ to be the *core* (i.e. *heart*) of the t-structure. By [BBD], $\mathcal{A}$ is an abelian category, in particular, every idempotent in $\mathcal{A}$ splits.

We will show that for each $n \geqslant 1$, every idempotent $e : X \to X$ with $w(X) \leqslant n$ splits. This will complete the proof. Use induction on the width. If $n = 1$, then $X \in \mathcal{A}[-i]$ with $i = b(X) = t(X)$. Since $\mathcal{A}$ and thus $\mathcal{A}[-i]$ are abelian categories, so e splits in $\mathcal{A}[-i]$, and thus in $\mathcal{D}$.

Assume now the assertion holds for n. Consider $e : X \to X$ to be an idempotent with $w(X) = n + 1$. Assume that $b(X) = m$. Therefore by Lemma 2.5, one has $\tau_{\leqslant m} X \in \mathcal{A}[-m]$ and $\tau_{\geqslant m+1} X \in \mathcal{D}^{\leqslant n+m} \cap \mathcal{D}^{\geqslant m+1}$, and thus $w(\tau_{\leqslant m} X) = 1$ and $w(\tau_{\geqslant m+1} X) \leqslant n$. Consider the following morphisms of exact triangles:

$$\begin{tikzcd}
\tau_{\leqslant m} X \arrow[r] \arrow[d, "\tau_{\leqslant m}(e)"] & X \arrow[r] \arrow[d, "e"] & \tau_{\geqslant m+1} X \arrow[r] \arrow[d, "\tau_{\geqslant m+1}(e)"] & (\tau_{\leqslant m} X)[1] \arrow[d, "\tau_{\leqslant m}(e)[1]"] \\
\tau_{\leqslant m} X \arrow[r] & X \arrow[r] & \tau_{\geqslant m+1} X \arrow[r] & (\tau_{\leqslant m} X)[1].
\end{tikzcd}$$

Note that both $\tau_{\leqslant m}(e)$ and $\tau_{\geqslant m+1}(e)$ are idempotents (by the functorial property of the truncation functors). By the induction hypothesis, both $\tau_{\leqslant m}(e)$ and $\tau_{\geqslant m+1}(e)$ split. Applying (TR2) and then Proposition 2.3, we obtain that e splits. This completes the proof. □

Acknowledgments

Let us remark that the theorem and its proof are inspired by some discussions with Prof. Michel Van den Bergh and Prof. Pu Zhang. We thank them very much.

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